Relativistic Hamiltonian guiding center drift formalism in anisotropic pressure magnetic coordinates

W. A. Cooper,[a)] J. P. Graves, and M. Jucker
Ecole Polytechnique Fédérale de Lausanne, Centre de Recherches en Physique des Plasmas, Association Euratom-Suisse, CH1015 Lausanne, Switzerland

M. Yu. Isaev
Nuclear Fusion Institute, RRC Kurchatov Institute, 123182 Moscow, Russia

(Received 8 June 2006; accepted 28 July 2006; published online 7 September 2006)

A Hamiltonian formulation of the relativistic guiding center drifts is extended to anisotropic pressure plasmas. The magnetic coordinates devised by Boozer are adapted to the anisotropic pressure model and retain canonical properties for two-dimensional and three-dimensional toroidal plasma equilibria including finite electrostatic perturbations provided that any electromagnetic perturbation only alters the parallel component of the vector potential. A mapping technique from arbitrary flux coordinates is outlined. A direct evaluation of the relativistic drift velocity recovers the equations of motion derived from the Hamiltonian formalism except for ignorable higher order terms in the evolution of the canonical angular variables and the effective parallel gyroradius.
 [DOI: 10.1063/1.2339025]

I. INTRODUCTION

In magnetic confinement systems with a strong magnetic field, the oscillation frequencies of the fields associated with macro and microinstabilities are small compared to the cyclotron species of the charged particles and their gyroradii are small compared with the characteristic scale lengths in the device. Under such conditions, it is computationally much more efficient and effective to follow the guiding centers rather than the exact particle orbits.[1,2] In the largest present-day tokamaks and stellarators, the contributions of energetic particles to β (the ratio of the kinetic to magnetic pressures) can be quite important. Typically in the Large Helical Device (LHD) heliotron, the hot particle fraction represents 1/3 of the total β in discharges with $\beta \geq 4\%$.[3] The LHD heliotron employs 10 MW of tangential neutral beam injection based on negative ion beam technology that produces energetic ions in the range of 150–180 keV. There are also strong indications that the pressure anisotropy induced by the energetic particles can be very significant in the LHD device.[4] Not only neutral beams, but also ion cyclotron resonance heating can distort the ion distribution function far from a Maxwellian state[5] to generate anisotropies in the total kinetic pressure. A relativistic formulation of the particle orbit problem is not generally justified for hot ions because their energies are almost exclusively well below 5 MeV rather than approaching the GeV range. However, the modelling of energetic electrons resulting from electron cyclotron heating (ECRH) and current drive (ECCD) which can exceed 100 keV in current experimental devices[6] would require a relativistic treatment. In low shear stellarator systems, it has been recognized that ECCD may be necessary to control the rotational transform not only to preclude low order rational surfaces from impacting magnetohydrodynamic (MHD) equilibrium and stability properties, but possibly more importantly to guarantee adequate rigidity for the implementation of magnetic island divertor design schemes. These issues provide the motivation for the formulation of the relativistic guiding center orbits within the background of a magnetically confined anisotropic plasma equilibrium state.

The application of Hamiltonian or Lagrangian formalisms to investigate guiding center drifts constitute the most compact, transparent and effective means to determine particle orbits.[7–18] Most of the work has concentrated on nonrelativistic particles in isotropic pressure plasmas in a set of magnetic coordinates introduced by Boozer[19] that can consider static magnetic fields with nested surfaces and arbitrary time dependent electric fields, however, any time dependent perturbed magnetic field is restricted to just twist and bend this field without compressing or stretching it.[8] These magnetic coordinates retain canonical properties under this constraint. For anisotropic pressure plasmas, White, Boozer and Hay applied a nonrelativistic drift Hamiltonian formulation in a set of canonical coordinates that are quite adequate but not employed in any of the principal guiding center/Monte Carlo δf computer codes in operation today.[10,20,21] One salient shortcoming of the coordinates in question is that periodicity around the torus is not trivially satisfied. Littlejohn applied these same canonical coordinates as the basis for the outline of a relativistic anisotropic Lagrangian formalism.[11] The relationship between these coordinates and the standard canonical coordinates used in the codes in isotropic nonrelativistic plasmas was derived by White and Chance.[10] They also presented a detailed derivation of the equations of motion from a Hamiltonian perspective. White subsequently adopted the Lagrangian formalism to explicitly derive the equations of motion in the isotropic nonrelativistic limit employing the standard canonical angular variables.[13]

Meiss and Hazeltine have proposed coordinate system that is canonical with arbitrary time dependent electrostatic

[a)]Electronic mail: wilfred.cooper@epfl.ch

and electromagnetic fields.[14] The canonical properties of the coordinates emerge because the radial components of both the electrostatic and electromagnetic fields in the covariant representation vanish. However, like in the Boozer framework, the transformation to these coordinates involves both the poloidal and toroidal angles, but results in a complicated nonlinear ordinary differential equation that severely handicaps their applicability.[18] The relativistic equations of motion in these generalized coordinates as well as in the standard magnetic coordinates in isotropic pressure plasmas have also been presented.[18] An alternative approach that retains the Hamiltonian properties with arbitrary perturbed fields has been developed by White and Zakharov,[22] but it is strictly valid only for axisymmetric systems. The canonical variables they have explored require the transformation of just the toroidal angle while maintaining the poloidal angle from the equilibrium computation fixed. This procedure has the added advantage that numerical resolution problems at the outside edge of each tokamak flux surface that are inherent in, for example, Boozer coordinates are avoided. For three-dimensional (3D) systems, the transformation outlined is only approximate and yields quasicanonical coordinates. Although the poloidal angle resolution is controlled, the corresponding resolution in the toroidal angle, unimportant in an axisymmetric tokamak, is not guaranteed in a stellarator.

In the present article, a relativistic Hamiltonian formulation of the guiding center drifts in the background of an anisotropic pressure equilibrium state with nested toroidal magnetic flux surfaces is developed. The magnetic coordinate system devised by Boozer[19] must be adapted to the anisotropic pressure conditions to obtain a canonical set of coordinates in which, as in the isotropic pressure limit, the perturbed magnetic fields are constrained to satisfy $\delta B_\parallel=0$, namely to only alter the component of the vector potential parallel to the equilibrium magnetic field lines. It should be stated that the model of White and Zakharov[22] can actually examine nonmagnetostatic conditions like pressure anisotropy also, but these are entirely concealed within the description of the equilibrium state. In the Boozer coordinate formulation we have adopted, the direct effects of the pressure anisotropy on the particle orbits through its impact on the equations of motion can be distinguished from the indirect effects on the equilibrium like the Shafranov shift, flux surface shaping (ellipticity, triangularity), etc. The mapping procedure from an arbitrary set of flux coordinates is outlined. Explicit expressions for the equations of motion of the canonical momenta and the canonical angular variables are described. In addition, the radial equation of motion and that of an effective parallel gyroradius are also derived. Verifications of these expressions are undertaken with direct evaluations of the relevant projections of the relativistic drift velocity subject to anisotropy in the pressure.

Considering the outline of this paper, the equilibrium state with finite pressure anisotropy is reviewed in Sec. II. The equilibrium state in magnetic coordinates (modified by the anisotropic pressure effects) is addressed in Sec. III together with the transformation from arbitrary flux coordinates. The relativistic drift Hamiltonian and the canonical momenta are presented in Sec. IV, while the Hamiltonian formalism to derive the equations of motion appears in Sec. V. The Summary and Conclusion are reported in Sec. VI. In the Appendix, the drift velocity for relativistic particles in plasmas with finite pressure anisotropy in a form that conserves the basic Hamiltonian properties of the system is considered and its projections are developed to directly verify the equations of motion obtained from the Hamiltonian formalism.

II. ANISOTROPIC PRESSURE MHD EQUILIBRIUM

The application of the Chew-Goldberger-Low (CGL) (Ref. 23) expansion of the pressure tensor as $p_\perp\mathbf{I}+(p_\parallel-p_\perp)\mathbf{bb}$, with $\mathbf{I}$ the unit tensor and $\mathbf{b}=\mathbf{B}/B$ the unit vector along the magnetic field lines, allows the MHD force balance relation that describes the equilibrium state for magnetically confined anisotropic pressure toroidal plasma systems to be expressed as[24]

$$\left.\frac{\partial p_\parallel}{\partial s}\right|_B \boldsymbol{\nabla} s=\boldsymbol{K}\times\boldsymbol{B}, \tag{1}$$

where $p_\parallel(s,B)$ represents the parallel pressure and is a functional of the radial variable s (typically chosen in the range $0\leq s\leq 1$) and the strength of the magnetic field B. The derivative of the parallel pressure with respect to s is evaluated at fixed B. It should be noted that in mirror systems where magnetic field line ends may be line tied, the pressures can also vary from field line to field line.

The effective parallel current density is defined as

$$\mu_0\boldsymbol{K}\equiv\boldsymbol{\nabla}\times(\sigma\boldsymbol{B}), \tag{2}$$

where $\mu_0=4\pi\times10^{-7}$ is the permeability of free space. We can extract the condition $\boldsymbol{K}\cdot\boldsymbol{\nabla}s=0$ from Eq. (1) which implies that the effective rather than the true current density lines lie on constant flux surfaces. The firehose stability criterion identified by the variable σ is given by the condition[24]

$$\sigma\equiv 1-\left.\frac{\mu_0}{B}\frac{\partial p_\parallel}{\partial B}\right|_s=1-\frac{\mu_0(p_\parallel-p_\perp)}{B^2}>0, \tag{3}$$

where $p_\perp(s,B)$ represents the perpendicular pressure. Another factor that enters into the description of the guiding center orbits is the mirror stability parameter[24]

$$\tau\equiv\left.\frac{\partial(\sigma B)}{\partial B}\right|_s=1+\left.\frac{\mu_0}{B}\frac{\partial p_\perp}{\partial B}\right|_s>0, \tag{4}$$

which guarantees that the equilibrium problem remains elliptic throughout the domain. The variables σ and τ are related to each other through the expression

$$\left(1+\left.\frac{B}{\sigma}\frac{\partial\sigma}{\partial B}\right|_s\right)=\frac{\tau}{\sigma}, \tag{5}$$

which will be repeatedly invoked to simplify the algebraic development of the drift equations of motion. It is also useful to note from Eq. (3) that the radial derivative of σ reduces to

$$\left.\frac{\partial\sigma}{\partial s}\right|_B=\frac{\mu_0}{B^2}\left(\left.\frac{\partial p_\perp}{\partial s}\right|_B-\left.\frac{\partial p_\parallel}{\partial s}\right|_B\right). \tag{6}$$

III. THE EQUILIBRIUM STATE IN MAGNETIC COORDINATES

A. Magnetic field representations

Maxwell's equation $\nabla\cdot\boldsymbol{B}=0$ and the condition $\boldsymbol{B}\cdot\nabla s=0$ that can be deduced from Eq. (1) yield a general description of the magnetic field in the contravariant representation as

$$\boldsymbol{B}=\psi'(s)\nabla v\times\nabla s+\Phi'(s)(\nabla s\times\nabla u+\nabla s\times\nabla\lambda), \qquad (7)$$

where u and v constitute the poloidal and toroidal angular variables, respectively, while the poloidal and toroidal magnetic flux functions are $\psi(s)$ and $\Phi(s)$, respectively. The symbol $(')$ denotes the derivative of a flux surface quantity with respect to s. The poloidal angle renormalization parameter λ,[25] which is a periodic function of u and v, can be split and combined with these angular variables to yield a coordinate system with straight magnetic field lines, namely

$$\boldsymbol{B}=\psi'(s)\nabla\varphi\times\nabla s+\Phi'(s)\nabla s\times\nabla\vartheta, \qquad (8)$$

where ϑ and φ are straight field line poloidal and toroidal angles.

The conditions $\nabla\cdot\boldsymbol{K}=\boldsymbol{K}\cdot\nabla s=0$ and the definition of $\boldsymbol{K}$ allow the expansion of the magnetic field in the covariant representation to acquire the general form

$$\sigma\boldsymbol{B}=\mu_0 J(s)\nabla u-\mu_0 I(s)\nabla v+\sigma b_s\nabla s+\nabla(\mu_0 Q). \qquad (9)$$

As with λ, we split Q and combine it with the angular variables to write the anisotropic version of the magnetic coordinates[19] and obtain the covariant form of the magnetic field as[8]

$$\sigma\boldsymbol{B}=\mu_0 J(s)\nabla\vartheta-\mu_0 I(s)\nabla\varphi+\sigma B_s\nabla s, \qquad (10)$$

where $I(s)$ and $J(s)$ are the poloidal and toroidal current flux functions, respectively. The radial magnetic field component $B_s(s,\vartheta,\varphi)$ is small and finite only for $\beta>0$ as demonstrated in the Appendix. From the expressions of the magnetic field in the covariant and contravariant representation, it is straightforward to derive the relation $\sigma\sqrt{g}B^2=\mu_0[\psi'(s)J(s)-\Phi'(s)I(s)]$, where $\sqrt{g}$ is the Jacobian of the transformation from Cartesian coordinates to the anisotropic Boozer frame. As discussed in Ref. 8 the factor μ_0/σ corresponds to the permeability of the system.

B. Mapping from equilibrium flux coordinates to magnetic coordinates

Magnetohydrodynamic equilibria are more efficiently computed numerically in flux coordinate systems that minimize the spectral width as demonstrated with the development of the VMEC code[25] in which the magnetic field is described by Eqs. (7) and (9). However, guiding center orbit, transport and MHD stability analysis are more transparent within the magnetic coordinate description originally devised by Boozer.[19] Therefore, a transformation to them from the equilibrium flux coordinates must be applied. For that purpose, we need to first determine the functions $\lambda(s,u,v)$ and $Q(s,u,v)$.

The poloidal angle renormalization parameter λ is actually calculated in the VMEC code.[25] It has been our experience that for the transformation to Boozer coordinates, the spectral width in VMEC is not always sufficiently broad for an accurate determination of λ. However, given an equilibrium state in arbitrary flux coordinates (including the VMEC coordinates), the Fourier amplitudes of the parameter λ can be computed numerically either by invoking the condition $\boldsymbol{K}\cdot\nabla s=0$ or defining the energy functional $\mathcal{W}(s)=\int_0^{2\pi/L}dv\int_0^{2\pi}du\sqrt{g_v}\sigma B^2/(2\mu_0)$ and applying the variation $\partial\mathcal{W}/\partial\lambda_\ell=0$ to obtain the matrix equation $\mathcal{M}_{\ell k}\lambda_k=-b_l$ on each flux tube. The subscript ℓ is the Fourier index for a mode pair (m_ℓ,n_ℓ) where m_ℓ is the poloidal mode number and n_ℓ is the toroidal mode number. The Jacobian of the transformation from the Cartesian to the equilibrium flux coordinates is $\sqrt{g_v}$ and L is the number of equilibrium field periods. The function Q is solved from the relations $\mu_0\partial Q/\partial u=\sigma B_u(s,u,v)-\mu_0 J(s)$ or $\mu_0\partial Q/\partial v=\sigma B_v(s,u,v)+\mu_0 I(s)$, where B_u and B_v are the covariant components of $\boldsymbol{B}$ in the covariant representation and are related to their contravariant counterparts using standard differential vector geometry. The flux surface variable s is invariant to coordinate transformation, but we can express the magnetic coordinate angles in terms of the flux coordinate angles used in the equilibrium computations as

$$\vartheta=u+\mathcal{G}(s,u,v), \qquad (11)$$

$$\varphi=v+\mathcal{L}(s,u,v), \qquad (12)$$

where $\mathcal{G}$ and $\mathcal{L}$ are periodic functions of the angular variables u and v. Substituting these expressions in Eqs. (8) and (10) and identifying the corresponding components of Eqs. (7) and (9) leads to a set of equations that can be inverted to obtain

$$\mathcal{G}(s,u,v)=\frac{\psi'(s)Q(s,u,v)-I(s)\Phi'(s)\lambda(s,u,v)}{\psi'(s)J(s)-\Phi'(s)I(s)}, \qquad (13)$$

$$\mathcal{L}(s,u,v)=\frac{\Phi'(s)[Q(s,u,v)-J(s)\lambda(s,u,v)]}{\psi'(s)J(s)-\Phi'(s)I(s)}. \qquad (14)$$

With this information, the Fourier amplitudes of any equilibrium quantity in the magnetic coordinate frame $\mathcal{H}(s,\vartheta,\varphi)$ can be computed from the equilibrium flux coordinate grid points, namely

$$\begin{aligned}\mathcal{H}_\ell(s)=&\frac{2L}{4\pi^2}\int_0^{2\pi/L}dv\int_0^{2\pi}du\mathcal{K}(s,u,v)\mathcal{H}(s,u,v)t_\ell\\&\times[m_\ell(u+\mathcal{G})-n_\ell L(v+\mathcal{L})]\\&-\delta_{m_\ell,0}\delta_{n_\ell,0}\langle\mathcal{K}(s,u,v)\mathcal{H}(s,u,v)\rangle,\end{aligned} \qquad (15)$$

where the kernel $\mathcal{K}(s,u,v)=(\sigma/\mu_0)\sqrt{g_v}B^2/[\psi'(s)J(s)-\Phi'(s)I(s)]$ and the symbol t_ℓ denotes a sine or cosine trigonometric function. $\langle\mathcal{A}\rangle$ represents the average integrated value of $\mathcal{A}$ over the angular variables and $\delta_{m_\ell,0}=0$ ($\delta_{m_\ell,0}=1$) for $m_\ell\neq 0$ ($m_\ell=0$).

IV. THE RELATIVISTIC DRIFT HAMILTONIAN AND THE CANONICAL MOMENTA

The relativistic drift Hamiltonian for guiding center particles is given by[11,17]

$$H_d = \gamma m_0 c^2 + e\chi(s,\vartheta,\varphi,t), \tag{16}$$

where e is the particle charge, m_0 is its rest mass, c is the speed of light, χ corresponds to the electrostatic potential and is a function of the spatial variables and time. The relativistic *gamma* factor can be written as

$$\gamma = \sqrt{1 + \frac{2\mu}{m_0 c^2}B + \frac{P_\parallel^2}{m_0^2 c^2}}, \tag{17}$$

where μ is the magnetic moment, a conserved quantity, and $P_\parallel$ is the momentum of a guiding center particle along the magnetic field lines of the equilibrium state. In the drift approximation, the canonical momentum is expressed as

$$\boldsymbol{P} = P_\parallel \boldsymbol{B}/B + e\boldsymbol{A}, \tag{18}$$

where $\boldsymbol{A}$ is the vector potential. Because the magnetic field strength corresponds to $\boldsymbol{\nabla}\times\boldsymbol{A}$, the equilibrium field contribution to the vector potential $\Phi(s)\boldsymbol{\nabla}\vartheta - \psi(s)\boldsymbol{\nabla}\varphi$ can be extracted from Eq. (8). However, the angular variables of the magnetic coordinates proposed by Boozer,[19] extended here to the anisotropic pressure model, retain a canonical structure when a magnetic perturbation only alters the parallel component of the vector potential, namely

$$\boldsymbol{A} = \Phi(s)\boldsymbol{\nabla}\vartheta - \psi(s)\boldsymbol{\nabla}\varphi + \Upsilon\sigma\boldsymbol{B}. \tag{19}$$

This constrained form for $\mathbf{A}$ corresponds to the condition $\delta B_\parallel = 0$ which is a valid approximation in low β plasmas for any perturbation. Arbitrary electromagnetic and electrostatic perturbations can be investigated in a different set of canonical coordinates,[14] but the mapping to them involves the solution of a complicated nonlinear ordinary differential equation.[18]

The canonical momenta in the covariant representation can then be written as

$$P_\vartheta = e\left\{\Phi(s) + \left[\frac{P_\parallel}{e\sigma B} + \Upsilon(s,\vartheta,\varphi,t)\right]\mu_0 J(s)\right\}, \tag{20}$$

$$P_\varphi = -e\left\{\psi(s) + \left[\frac{P_\parallel}{e\sigma B} + \Upsilon(s,\vartheta,\varphi,t)\right]\mu_0 I(s)\right\}, \tag{21}$$

where we have defined the canonical modified parallel gyroradius parameter ρ_c as

$$\rho_c \equiv \frac{P_\parallel}{e\sigma B} + \Upsilon(s,\vartheta,\varphi,t). \tag{22}$$

The expressions for the canonical momenta suggest that $s = s(P_\vartheta, P_\varphi)$ and $\rho_c = \rho_c(P_\vartheta, P_\varphi)$. Consequently, we can evaluate the derivatives of s and ρ_c with respect to the canonical momenta following the procedure outlined for nonrelativistic particles in scalar pressure plasmas.[10,13] These are required for the analysis of Hamilton's equations of motion.

V. HAMILTON'S EQUATIONS OF MOTION

Hamilton's canonical equations are

$$\dot{P_\vartheta} = -\left.\frac{\partial H_d}{\partial \vartheta}\right|_{P_\vartheta, P_\varphi, \varphi}, \qquad \dot{\vartheta} = \left.\frac{\partial H_d}{\partial P_\vartheta}\right|_{P_\varphi, \vartheta, \varphi}, \tag{23}$$

$$\dot{P_\varphi} = -\left.\frac{\partial H_d}{\partial \varphi}\right|_{P_\vartheta, P_\varphi, \vartheta}, \qquad \dot{\varphi} = \left.\frac{\partial H_d}{\partial P_\varphi}\right|_{P_\vartheta, \vartheta, \varphi}. \tag{24}$$

Noting that the Hamiltonian can be written as a function of ρ_c, B, Υ, χ and σ and applying the chain rule, we can derive the evolution equations for the canonical momenta as

$$\dot{P_\vartheta} = -\frac{e}{\gamma}\left(\frac{\mu}{e} + \sigma\tau\frac{eB}{m_0}\rho_\parallel^2\right)\frac{\partial B}{\partial \vartheta} - e\left[\left.\frac{\partial \chi}{\partial \vartheta}\right|_{s,\varphi,t} - \frac{e\sigma^2 B^2}{\gamma m_0}\rho_\parallel \left.\frac{\partial \Upsilon}{\partial \vartheta}\right|_{s,\varphi,t}\right], \tag{25}$$

$$\dot{P_\varphi} = -\frac{e}{\gamma}\left(\frac{\mu}{e} + \sigma\tau\frac{eB}{m_0}\rho_\parallel^2\right)\frac{\partial B}{\partial \varphi} - e\left[\left.\frac{\partial \chi}{\partial \varphi}\right|_{s,\vartheta,t} - \frac{e\sigma^2 B^2}{\gamma m_0}\rho_\parallel \left.\frac{\partial \Upsilon}{\partial \varphi}\right|_{s,\vartheta,t}\right], \tag{26}$$

where the subscripts indicate which variables are kept fixed during the evaluation of the partial derivatives. The evolution of the canonical angular variables reduces to

$$\dot{\vartheta} = -\frac{\mu_0 I(s)}{D}\left[\left.\frac{\partial \chi}{\partial s}\right|_{\vartheta,\varphi,t} + \frac{1}{\gamma}\left(\frac{\mu}{e} + \sigma\tau\frac{eB}{m_0}\rho_\parallel^2\right)\frac{\partial B}{\partial s} + \frac{eB^2\sigma}{\gamma m_0}\rho_\parallel^2\left.\frac{\partial \sigma}{\partial s}\right|_B\right] + \frac{e\sigma^2 B^2}{\gamma m_0 D}\rho_\parallel \times\left[\psi'(s) + (\rho_\parallel + \Upsilon)\mu_0 I'(s) + \mu_0 I(s)\left.\frac{\partial \Upsilon}{\partial s}\right|_{\vartheta,\varphi,t}\right], \tag{27}$$

$$\dot{\varphi} = -\frac{\mu_0 J(s)}{D}\left[\left.\frac{\partial \chi}{\partial s}\right|_{\vartheta,\varphi,t} + \frac{1}{\gamma}\left(\frac{\mu}{e} + \sigma\tau\frac{eB}{m_0}\rho_\parallel^2\right)\frac{\partial B}{\partial s} + \frac{eB^2\sigma}{\gamma m_0}\rho_\parallel^2\left.\frac{\partial \sigma}{\partial s}\right|_B\right] + \frac{e\sigma^2 B^2}{\gamma m_0 D}\rho_\parallel \times\left[\Phi'(s) + (\rho_\parallel + \Upsilon)\mu_0 J'(s) + \mu_0 J(s)\left.\frac{\partial \Upsilon}{\partial s}\right|_{\vartheta,\varphi,t}\right], \tag{28}$$

where we have defined the effective parallel gyroradius $\rho_\parallel \equiv P_\parallel/(e\sigma B) = \rho_c - \Upsilon$, we have invoked the relation described in Eq. (5) and the derivative of σ with respect to s is shown in Eq. (6). The choice of $\rho_\parallel$ in this form rather than the traditional expression $P_\parallel/(eB)$ simplifies the algebraic manipulations considerably. The factor D in the denominator corresponds to $D=\sigma\sqrt{g}B^2+\mu_0^2(\rho_\parallel+\Upsilon)[J(s)I'(s)-I(s)J'(s)]$. It is usually more convenient to follow the evolution of the radial position s and $\rho_\parallel$ of the guiding center particles rather than advancing the canonical momenta in time. We apply $\dot{s}=(\partial s/\partial P_\vartheta)\dot{P}_\vartheta+(\partial s/\partial P_\varphi)\dot{P}_\varphi$ to derive

$$\dot{s}=\frac{\mu_0 I(s)}{D}\left[\left.\frac{\partial\chi}{\partial\vartheta}\right|_{s,\varphi,t}+\frac{1}{\gamma}\left(\frac{\mu}{e}+\sigma\tau\frac{eB}{m_0}\rho_\parallel^2\right)\frac{\partial B}{\partial\vartheta}-\frac{e\sigma^2B^2}{\gamma m_0}\rho_\parallel\left.\frac{\partial\Upsilon}{\partial\vartheta}\right|_{s,\varphi,t}\right]+\frac{\mu_0 J(s)}{D}\times\left[\left.\frac{\partial\chi}{\partial\varphi}\right|_{s,\vartheta,t}+\frac{1}{\gamma}\left(\frac{\mu}{e}+\sigma\tau\frac{eB}{m_0}\rho_\parallel^2\right)\frac{\partial B}{\partial\varphi}-\frac{e\sigma^2B^2}{\gamma m_0}\rho_\parallel\left.\frac{\partial\Upsilon}{\partial\varphi}\right|_{s,\vartheta,t}\right]. \quad (29)$$

The derivation of the equation of motion for $\rho_\parallel$ follows a similar pattern to yield

$$\dot{\rho}_\parallel=-\left.\frac{\partial\Upsilon}{\partial t}\right|_{s,\vartheta,\varphi}-\frac{1}{D}\left[\psi'(s)+(\rho_\parallel+\Upsilon)\mu_0 I'(s)+\mu_0 I(s)\left.\frac{\partial\Upsilon}{\partial s}\right|_{\vartheta,\varphi,t}\right]\left[\left.\frac{\partial\chi}{\partial\vartheta}\right|_{s,\varphi,t}+\frac{1}{\gamma}\left(\frac{\mu}{e}+\sigma\tau\frac{eB}{m_0}\rho_\parallel^2\right)\frac{\partial B}{\partial\vartheta}\right]-\frac{1}{D}\left[\Phi'(s)+(\rho_\parallel+\Upsilon)\mu_0 J'(s)+\mu_0 J(s)\left.\frac{\partial\Upsilon}{\partial s}\right|_{\vartheta,\varphi,t}\right]\left[\left.\frac{\partial\chi}{\partial\varphi}\right|_{s,\vartheta,t}+\frac{1}{\gamma}\left(\frac{\mu}{e}+\sigma\tau\frac{eB}{m_0}\rho_\parallel^2\right)\frac{\partial B}{\partial\varphi}\right]+\frac{\mu_0}{D}\left[I(s)\left.\frac{\partial\Upsilon}{\partial\vartheta}\right|_{s,\varphi,t}+J(s)\left.\frac{\partial\Upsilon}{\partial\varphi}\right|_{s,\vartheta,t}\right]\left[\left.\frac{\partial\chi}{\partial s}\right|_{\vartheta,\varphi,t}+\frac{1}{\gamma}\left(\frac{\mu}{e}+\sigma\tau\frac{eB}{m_0}\rho_\parallel^2\right)\frac{\partial B}{\partial s}+\frac{eB^2\sigma}{\gamma m_0}\rho_\parallel^2\left.\frac{\partial\sigma}{\partial s}\right|_B\right]. \quad (30)$$

The anisotropy manifests itself through the functions σ and τ (and the radial derivative of σ) in the guiding center drift equations of motion, while the relativistic effects appear through the factor γ.

VI. SUMMARY AND CONCLUSIONS

A Hamiltonian formulation of the relativistic guiding center drifts has been applied to anisotropic pressure plasmas and the equations of motion have been explicitly derived in a Boozer magnetic coordinate system extended to treat conditions for which $p_\parallel \neq p_\perp$. The angular variables of these coordinates retain canonical properties for arbitrary but small electrostatic field perturbations and electromagnetic field perturbations constrained only to alter the vector potential component parallel to the equilibrium magnetic field. This model adequately represents the effects of bending and twisting of the magnetic fields on the particle orbits but fails to capture the impact of compression and stretching of these fields. Relativistic effects are only relevant and important for energetic electrons. By taking the limit $\gamma\to 1$, the corresponding nonrelativistic set of equations is recovered. The equations of motion derived are presented in the standard canonical variables employed in guiding center drift/Monte Carlo δf codes [ORBIT (Ref. 10), HAGIS (Ref. 20), VENUS (Ref. 21)] but extended to anisotropic pressure plasmas. In the limit $\sigma\to 1$ and $\tau\to 1$, the equations of motion for isotropic pressure plasmas are recovered.[18] The modifications to the Boozer magnetic coordinates induced by the pressure anisotropy have been presented and a Fourier mapping technique from arbitrary flux coordinates has been outlined. Finally, the equations of motion have been verified by a direct evaluation of the corresponding projections of the relativistic drift velocity (appropriately modified to conserve the Hamiltonian properties of the guiding center drift dynamics), provided that terms of higher order involving the product of the firehose stability parameter and the radial component of the magnetic field in the covariant representation are ignored.

The model developed constitutes a fundamental starting point for a consistent evaluation of energetic particle physics, in particular the impact of hot particles on transport, kinetic effects on MHD stability, microstability and turbulence.

ACKNOWLEDGMENT

This research was partially sponsored by the Fonds National Suisse de la Recherche Scientifique and Euratom.

APPENDIX: DIRECT VERIFICATION OF THE GUIDING CENTER DRIFTS

The guiding center drift velocity required in anisotropic pressure plasmas that retains the relevant components to second order in the gyroradius[2] and satisfies Liouville's theorem and conservation laws in systems with symmetry[8] corresponds to

$$\boldsymbol{v_d}=\frac{e\rho\sigma\{\boldsymbol{B}+\boldsymbol{\nabla}\times[(\rho_\parallel+\Upsilon)\sigma\boldsymbol{B}]\}}{\gamma m_0[1+(\rho_\parallel+\Upsilon)\langle\mu_0\boldsymbol{K}\cdot\boldsymbol{B}/B^2\rangle]}, \quad (A1)$$

where the effective parallel current density is

$$\frac{\mu_0 \boldsymbol{K}\cdot\boldsymbol{B}}{B^2} = \frac{\mu_0[J(s)I'(s) - I(s)J'(s)] + I(s)\frac{\partial(\sigma B_s)}{\partial\vartheta} + J(s)\frac{\partial(\sigma B_s)}{\partial\varphi}}{\psi'(s)J(s) - \Phi'(s)I(s)}. \tag{A2}$$

The integrated average of the effective parallel current density corresponds to

$$\left\langle \frac{\mu_0 \boldsymbol{K}\cdot\boldsymbol{B}}{B^2} \right\rangle \equiv \frac{L}{4\pi^2}\int_0^{2\pi/L} d\varphi \int_0^{2\pi} d\vartheta \frac{\mu_0 \boldsymbol{K}\cdot\boldsymbol{B}}{B^2}, \tag{A3}$$

where $L=1$ for an axisymmetric tokamak (the number of equilibrium field periods). We can immediately identify that the term $\sigma\sqrt{g}B^2[1+(\rho_\parallel+\Upsilon)\langle\mu_0\boldsymbol{K}\cdot\boldsymbol{B}/B^2\rangle]$ represents the function D defined in Sec. V. The guiding center drifts can be evaluated directly from $\boldsymbol{v_d}\cdot\boldsymbol{\nabla}s$, $\boldsymbol{v_d}\cdot\boldsymbol{\nabla}\vartheta$, $\boldsymbol{v_d}\cdot\boldsymbol{\nabla}\varphi$, and $\boldsymbol{v_d}\cdot\boldsymbol{\nabla}\rho_\parallel$.[7,12] The functional dependence $\rho_\parallel=\rho_\parallel(\chi,B,\sigma,t)$ facilitates the evaluation of the derivatives of $\rho_\parallel^2$ with respect to the magnetic coordinate variables. The radial component of the drift velocity recovers exactly the corresponding equation of motion derived from the Hamiltonian formalism [Eq. (29)]. The drift velocities in the poloidal and toroidal directions yield

$$\boldsymbol{v_d}\cdot\boldsymbol{\nabla}\vartheta = \dot{\vartheta} + \frac{e\sigma^2 B^2}{\gamma m_0 D}\rho_\parallel \frac{\partial[\sigma B_s(\rho_\parallel+\Upsilon)]}{\partial\varphi}, \tag{A4}$$

$$\boldsymbol{v_d}\cdot\boldsymbol{\nabla}\varphi = \dot{\varphi} - \frac{e\sigma^2 B^2}{\gamma m_0 D}\rho_\parallel \frac{\partial[\sigma B_s(\rho_\parallel+\Upsilon)]}{\partial\vartheta}, \tag{A5}$$

where $\dot{\vartheta}$ and $\dot{\varphi}$ correspond to the expressions obtained from the Hamiltonian formulation. The extra terms are formally of higher order $[O\sim(\rho\beta)]$ and are missing from the less exact Hamiltonian expressions in order to preserve the Hamiltonian dynamics. This can be seen by expanding the radial component of the equilibrium force balance relation [Eq. (1)] yields the expression

$$\sqrt{g}\boldsymbol{B}\cdot\boldsymbol{\nabla}(\sigma B_s) = \mu_0\sqrt{g}\frac{\partial p_\parallel}{\partial s}\bigg|_B - \mu_0\left\langle \sqrt{g}\frac{\partial p_\parallel}{\partial s}\bigg|_B \right\rangle. \tag{A6}$$

As the Jacobian $\propto 1/(\sigma B^2)$, the right-hand side roughly corresponds to the variation of the radial parallel pressure gradient divided by the magnetic energy density around a flux surface, which is small $(O\sim\beta)$. Similarly, the projection of the drift velocity normal to planes of constant $\rho_\parallel$ produces

$$\begin{aligned}\boldsymbol{v_d}\cdot\boldsymbol{\nabla}\rho_\parallel = \dot{\rho}_\parallel + \frac{\partial\Upsilon}{\partial t}\bigg|_{s,\vartheta,\varphi} - \frac{1}{D}\left[(\rho_\parallel+\Upsilon)\frac{\partial(\sigma B_s)}{\partial\varphi} + \sigma B_s\frac{\partial\Upsilon}{\partial\varphi}\bigg|_{s,\vartheta,t}\right]\left[\frac{\partial\chi}{\partial\vartheta}\bigg|_{s,\varphi,t} + \frac{1}{\gamma}\left(\frac{\mu}{e} + \frac{eB}{m_0}\sigma\tau\rho_\parallel^2\right)\frac{\partial B}{\partial\vartheta}\right] \\ + \frac{1}{D}\left[(\rho_\parallel+\Upsilon)\frac{\partial(\sigma B_s)}{\partial\vartheta} + \sigma B_s\frac{\partial\Upsilon}{\partial\vartheta}\bigg|_{s,\varphi,t}\right]\left[\frac{\partial\chi}{\partial\varphi}\bigg|_{s,\vartheta,t} + \frac{1}{\gamma}\left(\frac{\mu}{e} + \frac{eB}{m_0}\sigma\tau\rho_\parallel^2\right)\frac{\partial B}{\partial\varphi}\right]\end{aligned} \tag{A7}$$

which, noting that the derivative of Υ with respect to t is the negative of that of $\rho_\parallel$, recovers the leading order contributions from the Hamiltonian formalism and the extra terms all involve (σB_s), which is small.

[1] A. I. Morozov and L. S. Solov'ev, *Reviews of Plasma Physics*, edited by M. A. Leontovich (Consultants Bureau, New York, 1965), Vol. 2, p. 201.
[2] T. G. Northrop and J. A. Rome, Phys. Fluids **21**, 384 (1978).
[3] K. Y. Watanabe, S. Sakakibara, Y. Narushima, H. Funaba, K. Narihara, K. Tanaka, T. Yamaguchi, K. Toi, S. Ohdachi, O. Kaneko, H. Yamada, Y. Suzuki, W. A. Cooper, S. Murakami, N. Nakajima, I. Yamada, K. Kawahata, T. Tokuzawa, A. Komori, and the LHD experimental group, Nucl. Fusion **45**, 1247 (2005).
[4] T. Yamaguchi, K. Y. Watanabe, S. Sakakibara, Y. Narushima, K. Narihara, T. Tokuzawa, K. Tanaka, I. Yamada, M. Osakabe, H. Yamada, K. Kawahata, K. Yamazaki, and the LHD experimental group, Nucl. Fusion **45**, L33 (2005).
[5] R. W. Harvey, M. G. McCoy, G. D. Kerbel, and S. C. Chiu, Nucl. Fusion **26**, 43 (1986).
[6] S. Coda, S. Alberti, P. Blanchard, T. P. Goodman, M. A. Henderson, P. Nikkola, Y. Peysson, and O. Sauter, Nucl. Fusion **43**, 1361 (2003).
[7] A. H. Boozer, Phys. Fluids **23**, 904 (1980).
[8] R. B. White, A. H. Boozer, and R. Hay, Phys. Fluids **25**, 575 (1982).
[9] R. G. Littlejohn, Phys. Fluids **27**, 976 (1984).
[10] R. B. White and M. S. Chance, Phys. Fluids **27**, 2455 (1984).
[11] R. G. Littlejohn, Phys. Fluids **28**, 2015 (1985).
[12] J. Nuhrengerg and R. Zille, Phys. Lett. A **129**, 113 (1988).
[13] R. B. White, Phys. Fluids B **2**, 845 (1990).
[14] J. D. Meiss and R. D. Hazeltine, Phys. Fluids B **2**, 2563 (1990).
[15] W. Lotz, P. Merkel, J. Nührenberg, and E. Strumberger, Plasma Phys. Controlled Fusion **34**, 1037 (1992).
[16] R. B. White and A. H. Boozer, Phys. Plasmas **2**, 2915 (1995).
[17] A. H. Boozer, Phys. Plasmas **3**, 3297 (1996).
[18] W. A. Cooper, Plasma Phys. Controlled Fusion **39**, 931 (1997).
[19] A. H. Boozer, Phys. Fluids **25**, 520 (1982).
[20] S. D. Pinches, L. C. Appel, J. Candy, S. E. Sharapov, H. L. Berk, D. Borba, B. N. Breizman, T. C. Hender, K. I. Hopcraft, G. T. A. Huysmans, and W. Kerner, Comput. Phys. Commun. **111**, 133 (1998).
[21] O. Fischer, W. A. Cooper, and L. Villard, Nucl. Fusion **40**, 1453 (2000).
[22] R. White and L. E. Zakharov, Phys. Plasmas **10**, 573 (2003).
[23] G. F. Chew, M. L. Goldberger, and F. E. Low, Proc. R. Soc. London, Ser. A **236**, 112 (1956).
[24] H. Grad, Phys. Fluids **9**, 498 (1966).
[25] S. P. Hirshman and J. C. Whitson, Phys. Fluids **26**, 3553 (1983).