Exact canonical drift Hamiltonian formalism with pressure anisotropy and finite perturbed fields

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A Hamiltonian formulation of the guiding center drift orbits is extended to pressure anisotropy and field perturbations in axisymmetric systems. The Boozer magnetic coordinates are shown to retain canonical properties in anisotropic pressure plasmas with finite electrostatic perturbations and electromagnetic perturbed fields that solely affect the parallel component of the magnetic vector potential. The equations of motion developed in the Boozer coordinate frame are satisfied by direct verification of the drift velocities. A numerical application illustrates the significance of retaining all second order terms. © *2007 American Institute of Physics*. [DOI: 10.1063/1.2786061]

I. INTRODUCTION

When driven by the neutral beam injection or ion cyclotron resonance heating required to achieve high temperatures in magnetic confinement systems, energetic particles behave like independent test particles whose motion can be accurately tracked using the guiding center drift approximation. In addition, these energetic particles can generate a significant level of pressure anisotropy in the background equilibrium state.

The Lagrangian for particle drift motion constitutes an invaluable tool in determining canonical variables.[1,2] However, the canonical angular variables derived (in Refs. 1 and 2) do not satisfy toroidal periodicity conditions. White and Zakharov have developed a drift Hamiltonian formulation in generalized, nonstraight equilibrium magnetic field line coordinates that satisfies periodicity.[3] Yet, in practice this formulation appears to be highly cumbersome as most guiding center orbit codes are implemented in Boozer coordinates.

The principle issue that lies in the transformation to Boozer coordinates,[4] where periodicity is guaranteed, usually involves neglecting higher order terms.[2,5] However, Wang[6] has developed a technique for axisymmetric equilibrium fields in which the transformation to the Boozer coordinate frame retains all the relevant higher order terms neglected in previous treatments. The orbits can then be exactly identified with the drift velocity equation satisfying Liouville's theorem,[7]

$$\mathbf{V_d} \equiv \frac{e\rho_{\|}\sigma[\mathbf{B}+\nabla\times(\rho_c\sigma\mathbf{B})]}{m\left[1+\rho_c\mu_0\dfrac{\mathbf{K}\cdot\mathbf{B}}{B^2}\right]}, \quad (1)$$

where $\mu_0\mathbf{K}\equiv\nabla\times(\sigma\mathbf{B})$. In particular, the corrections seem to be of importance when the plasma is at high β and the particle has relatively large energy.

In this article, we extend the method developed by Wang to demonstrate that the Boozer magnetic coordinates also retain their canonical properties in equilibrium fields sustained with anisotropic pressure and containing both finite electrostatic and nearly incompressible electromagnetic field perturbations. The Lagrangian of the system is manipulated in Sec. II in order to define a set of canonical variables applicable to guiding center motion. In Sec. III, these variables form the basis of a Hamiltonian formulation, obtaining equations of motion that are then transformed to a canonical Boozer coordinate frame. A numerical application of the guiding center drifts is presented in Sec. IV, with a conversion to the notation of the VENUS code[8] outlined in the Appendix. Section V contains the conclusions and discussion.

II. LAGRANGIAN DETERMINATION OF CANONICAL VARIABLES

We begin by establishing, respectively, the covariant and contravariant representations of the equilibrium magnetic field and deriving its vector potential from the latter,

$$\sigma\mathbf{B}=g(\psi)\nabla\phi+I(\psi)\nabla\theta+g(\psi)\delta(\psi,\theta)\nabla\psi, \quad (2)$$

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$$\mathbf{B} = \nabla\phi \times \psi + q(\psi)\nabla\psi \times \theta = \nabla[\phi - q(\psi)\theta] \times \nabla\psi \tag{3}$$

$$= \nabla\psi \times \nabla\alpha = \nabla \times (\psi\nabla\alpha), \tag{4}$$

where $\mathbf{B}$ is the equilibrium magnetic field, (ψ, θ, ϕ) are the Boozer magnetic coordinates and $\alpha \equiv -\phi + q(\psi)\theta$. The basis for the difference in the representations of Eq. (2) between the anisotropic and isotropic pressure limits lies with the description of Ampère's law. For anisotropic pressure plasmas, Ampère's law is $\nabla \times \mathbf{H} = \mu_0 \mathbf{K}$, where $\mathbf{H} \equiv \sigma\mathbf{B}$ while in the isotropic limit (where $\sigma=1$), it is written as $\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$. The fields $\mathbf{K}$ and $\mathbf{j}$ correspond to the current density. Therefore, in the covariant Boozer representation, the magnetic field intensity $\mathbf{H}$ rather than its induction $\mathbf{B}$ is expressed in Eq. (2) under anisotropic pressure conditions. A valid interpretation is that the pressure anisotropy effectively modifies the permeability from that of free space μ_0 to μ_0/σ. We also find that the equilibrium vector potential of the magnetic field strength, $\mathbf{A_e}$, is $\psi\nabla\alpha$. The drift approximation of the momenta is defined as

$$\mathbf{P} = P_\parallel \frac{\mathbf{B}}{B} + e\mathbf{A}, \tag{5}$$

where the parallel momentum $P_\parallel \equiv m v_\parallel$, $v_\parallel$ is the particle velocity projection along the equilibrium field lines, while m and e correspond to the particle mass and charge, respectively. The vector potential consists of equilibrium and perturbed components ($\mathbf{A} = \mathbf{A_e} + \mathbf{A_p}$, where $\mathbf{A_p}$ is a finite perturbed function). We shall consider electromagnetic perturbations parallel to the equilibrium magnetic field only. Hence we define $\mathbf{A_p} \equiv V(\psi, \theta, \phi, t)\sigma\mathbf{B}$,[2,8–11] where $\sigma = 1 - \mu_0(p_\parallel - p_\perp)/B^2$ is the pressure anisotropy parameter with $p_\parallel$ and $p_\perp$ representing the parallel and perpendicular pressures, respectively.[5] Thus the Lagrangian can be specified, noting that $\rho_\parallel \equiv P_\parallel/(e\sigma B)$,

$$\mathcal{L}dt = \mathbf{P} \cdot \mathbf{dx} - \mathcal{H}dt \tag{6}$$

$$= e(\rho_c \sigma \mathbf{B} + \mathbf{A_e}) \cdot \mathbf{dx} - \mathcal{H}dt, \tag{7}$$

where ρ_c is defined as $\rho_\parallel + V$. By substituting α for ϕ in the covariant representation of the magnetic field, we determine

$$\sigma\mathbf{B} = h(\psi)\nabla\theta + g(\psi) \triangle(\psi,\theta)\nabla\psi - g(\psi)\nabla\alpha. \tag{8}$$

We have defined $\triangle(\psi,\theta)$ as $\delta(\psi,\theta) + q'(\psi)\theta$ and $h(\psi)$ as $I(\psi) + g(\psi)q(\psi)$, with the symbol ($'$) denoting the derivative of a flux surface quantity with respect to ψ. The Lagrangian becomes

$$\frac{\mathcal{L}dt}{e} = [\psi - \rho_c g(\psi)]d\alpha + \rho_c h(\psi)d\theta + \rho_c g(\psi) \triangle(\psi,\theta)d\psi - \frac{\mathcal{H}dt}{e}, \tag{9}$$

$$P_\alpha = \psi - \rho_c g(\psi). \tag{10}$$

We know that the Lagrangian must be of the form $\mathcal{L}dt = \Sigma_i p_i dq_i - \mathcal{H}dt$ in order to meet the conditions of a canonical coordinate system.[1] Thus it is necessary to eliminate the $d\psi$ term. This is facilitated by the introduction of the following:[6]

$$\alpha_c \equiv \alpha - \lambda(\psi, \theta, P_\alpha), \tag{11}$$

$$\lambda(\psi,\theta,P_\alpha) \equiv \int_{P_\alpha}^{\psi} \triangle(w,\theta)dw, \tag{12}$$

such that

$$d\alpha_c = d\alpha + \triangle(\psi,\theta)d\psi - \triangle(P_\alpha,\theta)dP_\alpha + \frac{\partial\lambda}{\partial\theta}d\theta, \tag{13}$$

$$\frac{\mathcal{L}dt}{e} = [\psi - \rho_c g(\psi)]d\alpha_c + \psi\triangle(\psi,\theta)d\psi + \left[\rho_c h(\psi) + P_\alpha \frac{\partial\lambda}{\partial\theta}\right]d\theta - P_\alpha \triangle(P_\alpha,\theta)dP_\alpha. \tag{14}$$

The equations of motion are invariant to the addition of a full time differential term to the Lagrangian.[1] Thus we introduce[6]

$$\mathcal{S} \equiv \int_{P_\alpha}^{\psi} w\triangle(w,\theta)dw, \tag{15}$$

$$\frac{\mathcal{L}dt}{e} - d\mathcal{S} = [\psi - \rho_c g(\psi)]d\alpha_c + \left[\rho_c h(\psi) + P_\alpha \frac{\partial\lambda}{\partial\theta} - \int_{P_\alpha}^{\psi} w \left.\frac{\partial\triangle}{\partial\theta}\right|_w dw\right]d\theta. \tag{16}$$

By setting $\partial_\theta \triangle(\psi,\theta) \equiv \partial_\psi Q(\psi,\theta)$,[6] we deduce that the poloidal component of the momentum in the covariant representation is

$$P_\theta = \rho_c h(\psi) + \rho_c g(\psi) Q(\psi,\theta) + \int_{P_\alpha}^{\psi} Q(w,\theta)dw. \tag{17}$$

We now reconsider ψ and ρ_c as functions of the canonical variables $(P_\theta, P_\alpha, \theta)$, where α_c is an ignorable variable. Thus from Eqs. (10) and (17), and setting $D \equiv g(\psi)q(\psi) + I(\psi) + g(\psi)I'(\psi) - \rho_c g'(\psi)I(\psi) - \rho_c g^2(\psi)\partial_\theta\delta(\psi,\theta)$, we calculate that

$$\frac{\partial\psi}{\partial P_\theta} = \frac{g(\psi)}{D}, \tag{18}$$

$$\frac{\partial\psi}{\partial P_\alpha} = \frac{g(\psi)}{D}\left[q(\psi) + Q(P_\alpha,\theta) - Q(\psi,\theta) + \frac{I(\psi)}{g(\psi)}\right], \tag{19}$$

$$\frac{\partial\psi}{\partial\theta} = \frac{g(\psi)}{D}\left[\rho_c g(\psi) \left.\frac{\partial Q}{\partial\theta}\right|_\psi - \int_{P_\alpha}^{\psi} \left.\frac{\partial Q}{\partial\theta}\right|_w dw\right], \tag{20}$$

$$\frac{\partial\rho_c}{\partial P_\theta} = \frac{1 - \rho_c g'(\psi)}{D}, \tag{21}$$

$$\frac{\partial\rho_c}{\partial P_\alpha} = \frac{1 - \rho_c g'(\psi)}{D}\left[q(\psi) + Q(P_\alpha,\theta) - Q(\psi,\theta) + \frac{I(\psi)}{g(\psi)}\right] - \frac{1}{g(\psi)}, \tag{22}$$

$$\frac{\partial \rho_c}{\partial \theta} = \frac{1-\rho_c g'(\psi)}{D}\left[\rho_c g(\psi)\left.\frac{\partial Q}{\partial \theta}\right|_{\psi} - \int_{P_\alpha}^{\psi}\left.\frac{\partial Q}{\partial \theta}\right|_{w} dw\right]. \tag{23}$$

Recalling that $\phi = q(\psi)\theta - \alpha_c - \lambda(\psi,\theta,P_\alpha)$, we also calculate the partial derivatives of the toroidal angle with respect to the canonical variables

$$\frac{\partial \phi}{\partial P_\theta} = -\frac{\partial \psi}{\partial P_\theta}\delta(\psi,\theta), \tag{24}$$

$$\frac{\partial \phi}{\partial P_\alpha} = -\frac{\partial \psi}{\partial P_\alpha}\delta(\psi,\theta) + \triangle(P_\alpha,\theta), \tag{25}$$

$$\frac{\partial \phi}{\partial \theta} = q(\psi) - \delta(\psi,\theta)\frac{\partial \psi}{\partial \theta} - Q(\psi,\theta) + Q(P_\alpha,\theta), \tag{26}$$

with the α_c differential of ϕ equal to the trivial negative unity.

III. HAMILTONIAN FORMULATION OF DRIFT ORBITS

The Hamiltonian associated with these canonical variables is actually $\mathcal{H}/e = e\rho_\parallel^2\sigma^2B^2/2m + \mu B/e + \chi \equiv \mathcal{H}_e$ where the equations of motion are defined as

$$\frac{dP_\theta}{dt} = -\left.\frac{\partial \mathcal{H}_e}{\partial \theta}\right|_{P_\theta,P_\alpha,\alpha_c}, \quad \frac{dP_\alpha}{dt} = -\left.\frac{\partial \mathcal{H}_e}{\partial \alpha_c}\right|_{P_\theta,P_\alpha,\theta},$$
$$\frac{d\theta}{dt} = \left.\frac{\partial \mathcal{H}_e}{\partial P_\theta}\right|_{P_\alpha,\theta,\alpha_c}, \quad \frac{d\alpha_c}{dt} = \left.\frac{\partial \mathcal{H}_e}{\partial P_\alpha}\right|_{P_\theta,\theta,\alpha_c}. \tag{27}$$

The variation of $\mathcal{H}_e$ with respect to the canonical variables is determined using Eqs. (18)–(27),

$$\frac{dP_\theta}{dt} = -D_\theta - \left[\frac{e\sigma^2B^2}{m}\rho_\parallel\frac{1-\rho_c g'(\psi)}{D} + \frac{g(\psi)}{D}[D_\psi - D_\phi\delta(\psi,\theta)]\right] \times\left[\rho_c g(\psi)\left.\frac{\partial Q}{\partial \theta}\right|_{\psi} - \int_{P_\alpha}^{\psi}\left.\frac{\partial Q}{\partial \theta}\right|_{w} dw\right] - D_\phi[q(\psi) + Q(P_\alpha,\theta) - Q(\psi,\theta)], \tag{28}$$

$$\frac{d\theta}{dt} = \frac{e\sigma^2B^2}{m}\rho_\parallel\frac{1-\rho_c g'(\psi)}{D} + \frac{g(\psi)}{D}[D_\psi - D_\phi\delta(\psi,\theta)], \tag{29}$$

$$\frac{dP_\alpha}{dt} = D_\phi, \tag{30}$$

$$\frac{d\alpha_c}{dt} = \left[\frac{e\sigma^2B^2}{m}\rho_\parallel\frac{1-\rho_c g'(\psi)}{D} + \frac{g(\psi)}{D}[D_\psi - D_\phi\delta(\psi,\theta)]\right] \times[q(\psi) + Q(P_\alpha,\theta) - Q(\psi,\theta)] - \frac{e\sigma^2B^2}{mD}\rho_\parallel\left[q(\psi) + \rho_c I'(\psi) - \rho_c g(\psi)\frac{\partial \delta}{\partial \theta}\right] + \frac{I(\psi)}{D}[D_\psi - D_\phi\delta(\psi,\theta)] + D_\phi\triangle(P_\alpha,\theta), \tag{31}$$

where the D_ψ, D_θ, and D_ϕ terms are equal to $D_{\psi_1}+D_{\psi_2}$, $D_{\theta_1}+D_{\theta_2}$, and $D_{\phi_1}+D_{\phi_2}$, respectively:

$$D_{\psi_1} = -\frac{e\sigma^2B^2}{m}\rho_\parallel\frac{\partial V}{\partial \psi}, \tag{32}$$

$$D_{\psi_2} = \left(\frac{e\tau\sigma B}{m}\rho_\parallel^2 + \frac{\mu}{e}\right)\frac{\partial B}{\partial \psi} + \frac{e\sigma B^2}{m}\rho_\parallel^2\frac{\partial \sigma}{\partial \psi} + \frac{\partial \chi}{\partial \psi}, \tag{33}$$

$$D_{\theta_1} = -\frac{e\sigma^2B^2}{m}\rho_\parallel\frac{\partial V}{\partial \theta}, \tag{34}$$

$$D_{\theta_2} = \left(\frac{e\tau\sigma B}{m}\rho_\parallel^2 + \frac{\mu}{e}\right)\frac{\partial B}{\partial \theta} + \frac{\partial \chi}{\partial \theta}, \tag{35}$$

$$D_{\phi_1} = -\frac{e\sigma^2B^2}{m}\rho_\parallel\frac{\partial V}{\partial \phi}, \tag{36}$$

$$D_{\phi_2} = \frac{\partial \chi}{\partial \phi}. \tag{37}$$

Rather than following the momenta and the coordinate α_c (which does not trivially satisfy periodicity), it is both more convenient and physically intuitive to evaluate the particle motion in the Boozer coordinate frame and the parallel gyroradius.[1] By expanding the time differential of $\psi(P_\theta,P_\alpha,\theta)$ and calculating the time differential of $\phi = q(\psi)\theta - \alpha$ we find that

$$\frac{d\psi}{dt} = \frac{I(\psi)}{D}D_\phi - \frac{g(\psi)}{D}D_\theta, \tag{38}$$

$$\frac{d\phi}{dt} = \frac{e\sigma^2B^2}{m}\rho_\parallel\frac{q(\psi) + \rho_c I'(\psi) - \rho_c g(\psi)\frac{\partial \delta}{\partial \theta}}{D} + \frac{g(\psi)\delta(\psi,\theta)}{D}D_\theta - \frac{I(\psi)}{D}D_\psi. \tag{39}$$

Furthermore we calculate the motion equation along $\rho_\parallel$ by subtracting the time differential of $V(\psi,\theta,\phi,t)$ from that of $\rho_c(P_\theta,P_\alpha,\theta)$,

$$\frac{d\rho_c}{dt} = -\frac{1-\rho_c g'(\psi)}{D}D_\theta - \frac{q(\psi) + \rho_\parallel I'(\psi) - \rho_\parallel g(\psi)\frac{\partial \delta}{\partial \theta}}{D}D_\phi, \tag{40}$$

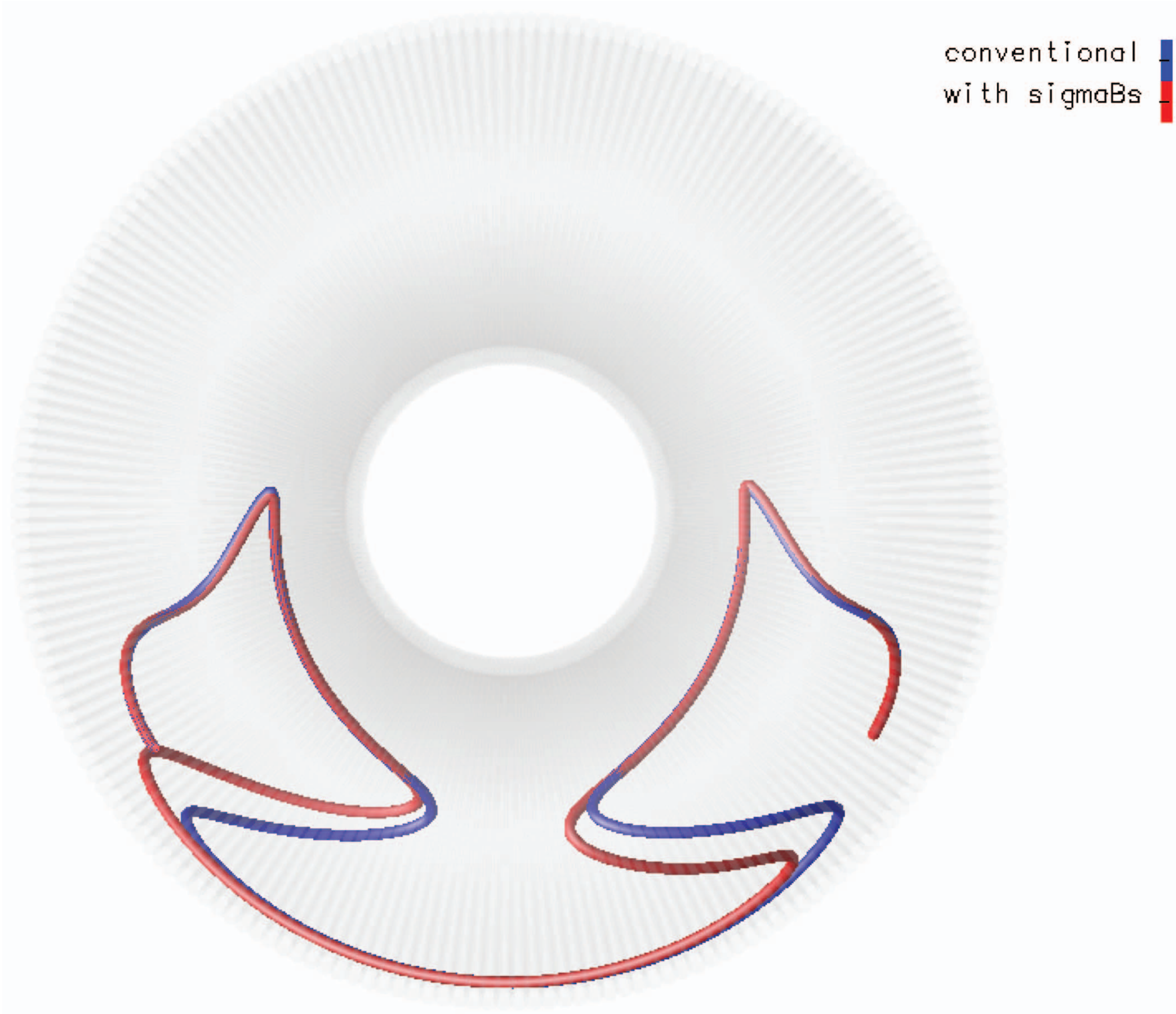

FIG. 1. (Color) A bird's eye view of the torus showing the orbits of a trapped particle commencing at the vertical midplane. The blue trajectory corresponds to the standard formulation of guiding center motion, while the red orbit is integrated using Eqs. (43)–(46) with perturbed fields neglected. Under these conditions, the inclusion of full second order components solely affects the toroidal drift. This particular perspective gives a clear presentation of the toroidal displacement of the two orbits.

$$\frac{dV}{dt} = -\frac{1-\rho_c g'(\psi)}{D}D_{\theta_1} - \frac{q(\psi)+\rho_\parallel I'(\psi)-\rho_\parallel g(\psi)\dfrac{\partial\delta}{\partial\theta}}{D}D_{\phi_1} + \frac{g(\psi)}{D}\left[\frac{\partial V}{\partial\theta}D_{\psi_2} - \frac{\partial V}{\partial\psi}D_{\theta_2} + \delta(\psi,\theta)\left(\frac{\partial V}{\partial\phi}D_{\theta_2} - \frac{\partial V}{\partial\theta}D_{\phi_2}\right)\right] + \frac{I(\psi)}{D}\left[\frac{\partial V}{\partial\psi}D_{\phi_2} - \frac{\partial V}{\partial\phi}D_{\psi_2}\right] + \frac{\partial V}{\partial t}, \quad (41)$$

$$\frac{d\rho_\parallel}{dt} = \frac{I(\psi)\dfrac{\partial V}{\partial\phi} - g(\psi)\dfrac{\partial V}{\partial\theta}}{D}D_{\psi_2} + \frac{g(\psi)\dfrac{\partial V}{\partial\psi} - g(\psi)\delta(\psi,\theta)\dfrac{\partial V}{\partial\phi} - 1 + \rho_c g'(\psi)}{D}D_{\theta_2} + \frac{g(\psi)\delta(\psi,\theta)\dfrac{\partial V}{\partial\theta} - I(\psi)\dfrac{\partial V}{\partial\psi} - q(\psi) - \rho_c I'(\psi) + \rho_c g(\psi)\dfrac{\partial\delta}{\partial\theta}}{D}D_{\phi_2} - \frac{\partial V}{\partial t}. \quad (42)$$

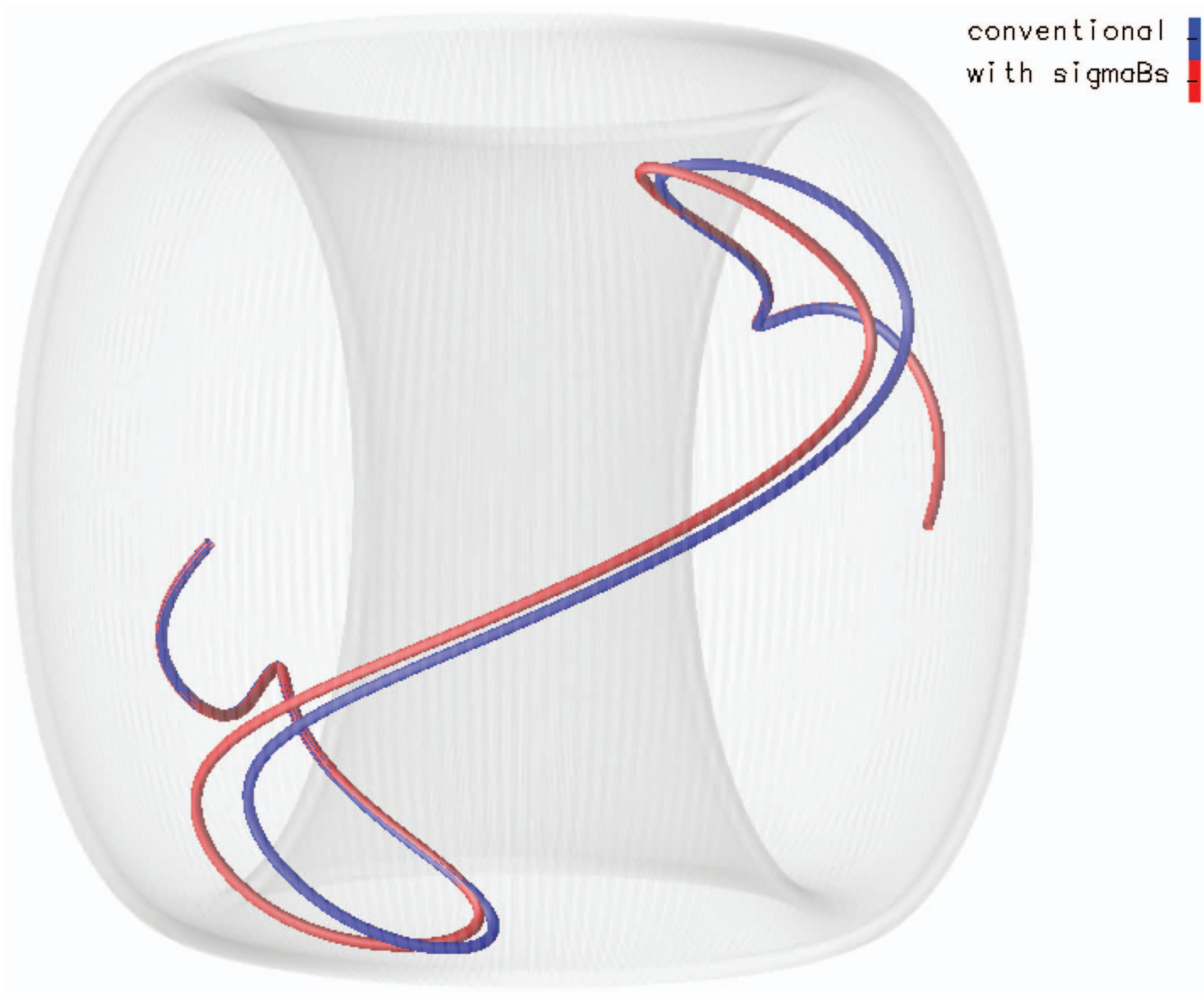

FIG. 2. (Color) A full toroidal view illustrates another perspective of the toroidal displacement of the two orbits. The blue trajectory corresponds to the standard formulation of the guiding center motion, while the red orbit is integrated using Eqs. (43)–(46) with perturbed fields neglected. Under these conditions, the inclusion of full second order components solely affects the toroidal drift.

An evaluation of the projections of the drift velocity, $\mathbf{V_d}$, with respect to ψ, θ, ϕ, and $\rho_\parallel$ recovers precisely the Hamiltonian formulation of the Boozer coordinate equations of motion calculated and expressed in Eqs. (29), (38), (39), and (42). The drift velocity that satisfies Liouville's theorem is given by Eq. (1).[7] It can be demonstrated that $\dot{\psi}=\mathbf{V_d}\cdot\nabla\psi$, $\dot{\theta}=\mathbf{V_d}\cdot\nabla\theta$, $\dot{\phi}=\mathbf{V_d}\cdot\nabla\phi$, and $\dot{\rho}_\parallel=\partial_t\rho_\parallel+\mathbf{V_d}\cdot\nabla\rho_\parallel$, where $\partial_t\rho_\parallel=-\partial_t V$.

IV. APPLICATION WITH THE VENUS CODE

In order to implement Eqs. (29), (38), (39), and (42) into the VENUS code,[8,5] they must first be converted to the notation employed in the program. When the required transformations are made (see the Appendix), the equations of motion become

$$\frac{ds}{dt}=\frac{\mu_0\mathcal{J}(s)}{D_v}D_\phi+\frac{\mu_0\mathcal{I}(s)}{D_v}D_\theta, \tag{43}$$

$$\frac{d\theta}{dt}=\frac{e\sigma^2B^2}{m}\rho_\parallel\frac{\psi'(s)+\rho_c\mu_0\mathcal{I}'(s)}{D_v}-\frac{\mu_0\mathcal{I}(s)}{D_v}D_s-\frac{D_\phi}{D_v}\sigma B_s, \tag{44}$$

$$\frac{d\phi}{dt}=\frac{e\sigma^2B^2}{m}\rho_\parallel\frac{\Phi'(s)+\rho_c\mu_0\mathcal{J}'(s)}{D_v}-\frac{\mu_0\mathcal{J}(s)}{D_v}D_s+\frac{D_\theta}{D_v}\sigma B_s-\frac{e\sigma^2B^2}{m}\rho_\parallel\frac{\rho_c}{D_v}\frac{\partial(\sigma B_s)}{\partial\theta}, \tag{45}$$

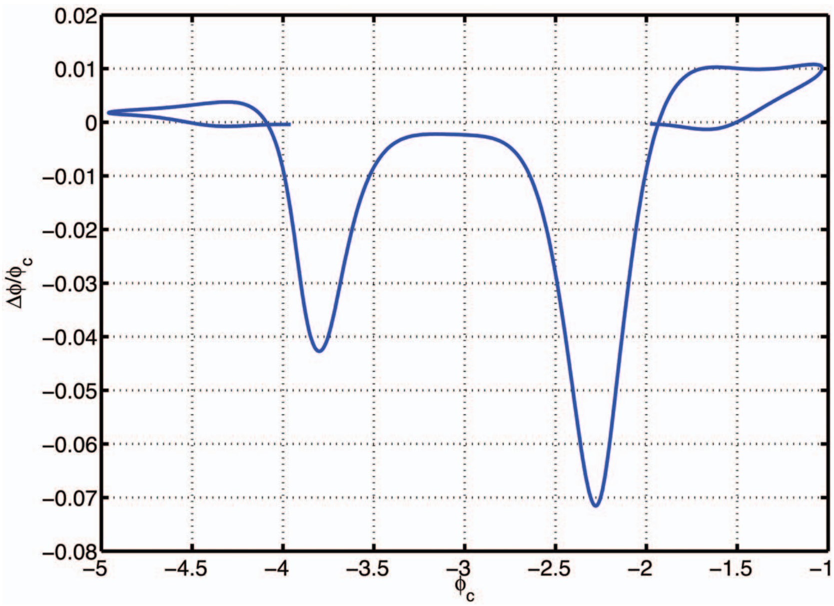

FIG. 3. (Color) A graph of the difference between the toroidal drifts with respect to ϕ_c (the toroidal drift using the conventional formulation of the orbit equation) illustrates the deviation that arises when certain second order terms are ignored. The difference is calculated as ϕ_c subtracted by ϕ_n, where ϕ_n is the toroidal drift motion calculated using Eq. (45) with perturbed fields neglected.

$$\frac{d\rho_\|}{dt} = \frac{\mu_0\left[\mathcal{J}(s)\frac{\partial V}{\partial \phi} + \mathcal{I}(s)\frac{\partial V}{\partial \theta}\right]}{D_v} D_{s_2} - \frac{\psi'(s) + \mu_0\mathcal{I}(s)\frac{\partial V}{\partial s} + \rho_c\mu_0\mathcal{I}'(s)}{D_v} D_{\theta_2} - \frac{\mu_0\mathcal{J}(s)\frac{\partial V}{\partial s} + \Phi'(s) + \rho_c\mu_0\mathcal{J}'(s)}{D_v} D_{\phi_2} - \frac{\partial V}{\partial t} + \left[\frac{\frac{\partial V}{\partial \theta}D_{\phi_2}}{D_v} - \frac{\frac{\partial V}{\partial \phi}D_{\theta_2}}{D_v}\right]\sigma B_s + \frac{\rho_c D_{\phi_2}}{D_v}\frac{\partial(\sigma B_s)}{\partial \theta}. \quad (46)$$

The terms containing σB_s, in some form or another, were factored out because they are essentially those that have not previously been included in the VENUS code. These terms are all of second order and while previous treatments contained some terms of this order, this formulation is significant as it contains all terms of $\sim O(\beta\rho_\|)$.

In order to demonstrate the significance of retaining all second order terms, the equations of motion were stripped of all perturbations before being implemented into the code. In the absence of perturbed fields, the full second order drift contributions to the guiding center orbits only alters the equation of motion for the toroidal drift. Our numerical application with the VENUS code was considered for a 10 MeV trapped proton orbit at a high β of 19%. In addition, the equilibrium magnetic field was obtained from the VMEC code[12] with a central value of 5.6 T. We estimate that a 3.5 MeV α particle in a 6.6 T field would recover the same difference in orbits as that investigated.

The toroidal displacement is best observed from the perspective of Fig. 1. The orbit described by the red curve has been integrated using Eqs. (43)–(46) while the blue curve represents the standard formulation of these equations of motion in which certain higher order terms are neglected. The maximum deviation between the two orbits occurs about halfway between the vertical midplane and the turning points of the orbits. A full toroidal view is presented in Fig. 2. Figure 3 is a graph of the difference of the two toroidal drifts with respect to the conventional toroidal drift in which certain second order terms are ignored. The importance of including all second order terms is demonstrated as the percent difference plotted here reaches up to 7%.

V. DISCUSSION

The method developed by Wang[6] is extended to demonstrate that Boozer coordinates are also canonical for guiding center drift orbit motion in the presence of both arbitrary electrostatic perturbed fields and electromagnetic perturbations with vector potential parallel to the equilibrium magnetic field. The formulation is further extended to conditions where energetic particles drive an anisotropic pressure plasma. The equations of motion we have derived in our Hamiltonian formalism are recovered exactly from direct verification of the corresponding projections of the guiding center drift velocity. In addition, a numerical application in the VENUS code was presented to illustrate the significance of retaining all second order terms.

The introduction of $\mathcal{S}dt$ into our Lagrangian eliminates the $d\psi$ term and establishes α_c and θ as our canonical coordinate variables. The Hamiltonian derivation of these coordinates and their respective momenta develops a set of equations of motion that, while canonical, do not satisfy other criteria (such as periodicity). Importantly, when the required conversion to the Boozer reference frame is made the equations of motion retain their canonical nature, facilitating the implementation of numerical schemes.

It is to be noted that our formulation of the perturbed vector potential rests on the assumption (made extensively in existing codes) of a relatively low ratio of kinetic to magnetic pressures, β.[2,8–11] Furthermore, the formulation is calculated in axisymmetry and thus is only applicable to tokamak and reverse field pinch conditions.

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APPENDIX: CONVERSION TO VENUS NOTATION

The following transformations were made to Eqs. (29), (38), (39), and (42):

$$g(\psi) \Rightarrow -\mu_0 \mathcal{I}(\psi),$$

$$I(\psi) \Rightarrow \mu_0 \mathcal{J}(\psi),$$

$$g(\psi)\delta(\psi,\theta) \Rightarrow \sigma B_\psi,$$

$$g(\psi)q(\psi) + I(\psi) \Rightarrow \sigma\sqrt{g}B^2.$$

The VENUS code does not include ψ as a coordinate variable. Instead the code is written in terms of s, where $\psi = \psi(s)$. The term s is usually proportional to the plasma volume enclosed and varies from 0 at the magnetic axis to unity at the plasma-vacuum interface. The following transformations were made with γ representing any flux surface quantity,

$$\gamma(\psi) \Rightarrow \gamma(s),$$

$$\gamma'(\psi) \Rightarrow \frac{\gamma'(s)}{\psi'(s)}.$$

Also,

$$\sigma B_\psi \Rightarrow \frac{\sigma B_s}{\psi'(s)}.$$

Note that the D_ψ term becomes D_s where $D_\psi \Rightarrow \psi'(s)D_s$. Furthermore, our Jacobian in the VENUS notation becomes $\sqrt{g} = \mu_0[\mathcal{J} - \mathcal{I}q(\psi)]/\sigma B^2$. However the Jacobian of the VENUS code is in fact $\sqrt{g_v} = \mu_0[\psi'(s)\mathcal{J}(s) - \Phi'(s)\mathcal{I}(s)]/\sigma B^2$, where $\Phi'(s) \equiv q(\psi)\psi'(s)$. Thus,

$$D \Rightarrow \frac{D_v}{\psi'(s)}.$$

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